Tannaka-Krein Duality and the Complexification of Compact Lie Groups

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1 Introduction

Take a finite abelian locally compact group G. From this the set of characters of this group is the set of homomorphisms $\mathrm{Hom}(G, \mathbb{T})$, from the group to the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. This set has a special name, the Pontryagin dual, and is denoted by $\check{G}$. Given this set of characters, to what extend can we determine the original group G? That is to say, if we know $\check{G}$, can we deduce the original group G?

The answer turns out to be completely, and is a special case of this Tannaka-Krien duality. To see how this arise we can first put a group structure on $\check{G}$, defined by pointwise multiplication. We can define an isomorphism $G \to \check{G}$, however it's a noncanonical one. So the question now becomes can we get a canonical isomorphism between G and some object? The answer comes from looking at the double duel, $\check{\check{G}}$. If we take this then a theorem of Pontryagin asserts the following more general statement:

Theorem (Pontryagin Duality)**.** Let G an abelian (not necessarily finite) locally compact group

$$G \cong \check{\check{G}}$$
$$g \mapsto (\chi \mapsto \chi(g))$$

To this answers our question, for a finite abelian locally compact group G the group of characters of G can be used to determine G itself.

What about the nonabelian case? Here we cannot use the group of characters, as two nonisomorphic nonabelian groups may give rise to the same character table. An example of this is for the groups D_8 and Q_8, which are nonisomorphic, but it is know give the same table. So now we ask, what set of objects can give back the original group G in the nonabelian case?

We've seen in class how the individual entries of a matrix representation of a Lie group G are continuous functions on G. These matrix representations generate the ring of representative functions. Part of the Tannaka-Krien duality states that this ring of representative functions defines the structure of G. Moreover this ring can define a complexification of a compact Lie group.

Tannaka-Krien Duality concerns reconstructing a compact Lie group from the algebra of representative functions. This theorem can be stated as the rescontruction from the category of representations of a compact Lie group, but the focus for us will be on the former.

2 Background

Let $C^0(G, K)$ be the ring of continuous functions $G \to K$ for a field K. Throughout this project $K = \mathbb{R}$ or $\mathbb{C}$. From class we know that left and right translations yield actions of G on this ring, for example the right translation is

$$R : G \times C^0(G, K) \to C^0(G, K)$$
$$R(g, f)(x) = f(xg)$$

From here we define a representative function of a Lie group G

Definition. Let G act on $C^0(G, K)$ via this right action. An element of the ring is called a representative function if it generates a finite dimensional G-subspace of the ring $C^0(G, K)$

Proposition. The representative functions form a K-subalgebra of $C^0(G, K)$, denoted $\mathcal{F}(G, K)$

Proof. Take two matrix representatives $g \mapsto (f_{ij}(g))$ and $g \mapsto (r_{mn}(g))$. Then the direct sum and tensor products of these representative functions yield that the sum and product of f_{ij} and r_{mn} are also representative functions. Hence this gives the structure of a K-subalgebra. □

A theorem of Peter and Weyl shows that $\mathcal{F}(G, K)$ is in fact dense inside $C^0(G, K)$

3 Tannaka-Krien Duality

For this section we take the field K to be $\mathbb{R}$. As stated above Tannaka-Krien demonstrated that if we know $\mathcal{F}(G,\mathbb{R})$ then we can reconstructed the compact Lie group G back from it. We do this specifically by looking at all $\mathbb{R}$-algebra homomorphisms from the algebra $\mathcal{F}(G,\mathbb{R})$ to $\mathbb{R}$ and show that this set is in fact isomorphic to the compact Lie group G. Let $G_\mathbb{R}$ be this set.

Each $g \in G$ yields a map in $G_\mathbb{R}$ via evaluation on the representative function. i.e.

$$e_g : \mathcal{F}(G,\mathbb{R}) \to \mathbb{R} \quad f \mapsto f(g)$$

Thus we get a map $i : G \to G_\mathbb{R}$ which sends $g \mapsto e_g$. This is the map that we will eventually show is an isomorphism of compact Lie groups. To get there we proceed in stages: First showing that $G_\mathbb{R}$ is a group, then we'll define a topology on it to make it a topological group. Finally we show that it's in fact a compact Lie group.

We begin with the group properties of $G_\mathbb{R}$. For this we generalize to G_K for any field, and require that G_K be equipped with a multiplication, identity and inverse that satisfy the usual group axioms. First a lemma:

Lemma. The K-algebra homomorphism

$$t : \mathcal{F}(G,K) \otimes_K \mathcal{F}(H,K) \to \mathcal{F}(G \times H, K)$$

Which sends

$$(u \otimes v) \mapsto ((g,h) \mapsto u(g)v(h))$$

Is an isomorphism

Proof. We show that this map is a bijection. To show surjectivity let $f \in \mathcal{F}(G \times H, K)$ be given, and $S \subset \mathcal{F}(H,K)$ be generated by the functions

$$h \mapsto f(g,h)$$

As this space is finite dimensional there is a basis $e_1, ..., e_n$ such that there are elemetns $h_1, ..., h_n \in H$ for which $e_i(h_j) = \delta_{ij}$. Write

$$f(g,h) = \sum_i u_i(g) e_i(h)$$

Then we have that $u_i(g) = f(g, h_i)$, so $u_i \in \mathcal{F}(G,K)$

To show injectivity we again have there is a basis $e_1, ..., e_n$ such that there are elemetns $h_1, ..., h_n \in H$ for which $e_i(h_j) = \delta_{ij}$. Let $f \in \mathcal{F}(G,K) \otimes \mathcal{F}(H,K)$ be such that $f \in \ker(t)$. Then we can write we can find a basis as above, for which

$$f = \sum_i u_i \otimes e_i$$

and each $u_i \in \mathcal{F}(G,K)$. Then

$$u_i(g) = \sum_i u_i(g)e_i(h_j) = 0$$

Hence $f = 0$. □

3.1 A special algebraic structure

We want to use $\mathcal{F}(G,K)$ as a kind of model for the Lie group G. To do this we have to translate statements and axioms of groups into those of algebras. We begin with group multiplication, which induces a homomorphism

$$\mathcal{F}(G,K) \to \mathcal{F}(G \times G, K) \quad f \mapsto ((g,h) \mapsto f(gh))$$

From here we get a K-algebra homomorphism

$$d : \mathcal{F}(G,K) \to \mathcal{F}(G \times G, K) \cong \mathcal{F}(G,K) \otimes \mathcal{F}(G,K)$$

Which is called comultiplication. We also get an induced K-algebra homomorphism from the inverse map $g \mapsto g^{-1}$ denoted c which works as follows:

$$c : \mathcal{F}(G,K) \to \mathcal{F}(G,K) \quad c(f)(g) \mapsto f(g^{-1})$$

Much like the induced multiplication map was called comultiplication, we call c the coinverse. Finally we have a counit

$$\epsilon : \mathcal{F}(G,K) \to K$$

As a little aside recall we can define a K-algebra as a triple (A, m, i) where m, i are the multiplication and inverse maps defined as follows:

$$m : A \otimes A \to A \quad i : K \to A$$

The 'co'-prefix refers to the fact that we reverse the arrows in the standard diagrams. The following are a translation of the axioms of a group.

- The coassociativity of d satisfies

  $$(d \otimes id) \circ d = (id \otimes d) \circ d$$

  So the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(G,K) & \xrightarrow{d} & \mathcal{F}(G,K) \otimes \mathcal{F}(G,K) \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle id \otimes d} \\ \mathcal{F}(G,K) \otimes \mathcal{F}(G,K) & \xrightarrow{d \otimes id} & \mathcal{F}(G,K) \otimes \mathcal{F}(G,K) \otimes \mathcal{F}(G,K) \end{array}$$

- The counit satisfies

$$(\epsilon \otimes id) \circ d = id = (id \otimes \epsilon) \circ d$$

So the following diagram commutes

$$\begin{array}{ccccc}
\mathcal{F}(G,K) & \xrightarrow{\quad d \quad} & & & \mathcal{F}(G,K) \otimes \mathcal{F}(G,K) \\
\downarrow d & & \searrow^{id} & & \downarrow id\otimes\epsilon \\
\mathcal{F}(G,K) \otimes \mathcal{F}(G,K) & \xrightarrow{\epsilon\otimes id} & K \otimes \mathcal{F}(G,K) \cong \mathcal{F}(G,K) & \cong & K \otimes \mathcal{F}(G,K)
\end{array}$$

- The coinverse satisfies

$$m \circ (c \otimes id) \circ d = i \circ \epsilon$$

Where m, i are the maps above. In a commutative diagram this is:

$$\begin{array}{ccccc}
\mathcal{F}(G,K) \otimes \mathcal{F}(G,K) & \xrightarrow{m} & \mathcal{F}(G,K) & \xleftarrow[m]{} & \mathcal{F}(G,K) \otimes \mathcal{F}(G,K) \\
{\scriptstyle id\otimes c}\uparrow & & {\scriptstyle i\circ\epsilon}\uparrow & & {\scriptstyle c\otimes id}\uparrow \\
\mathcal{F}(G,K) \otimes \mathcal{F}(G,K) & \xleftarrow[d]{} & \mathcal{F}(G,K) & \xrightarrow{d} & \mathcal{F}(G,K) \otimes \mathcal{F}(G,K)
\end{array}$$

The algebra $\mathcal{F}(G,K)$ along with d, c and ϵ are what is known as a Hopf algebra.

3.2 Group Structure

All this to say that we use the above to define a group multiplication on G_K via a composition. Let $s, t \in G_K$ then $s \cdot t$ is defined as follows

$$s \cdot t : \mathcal{F}(G,K) \xrightarrow{d} \mathcal{F}(G,K) \otimes_K \mathcal{F}(G,K) \xrightarrow{s\otimes t} K \otimes_K K \cong K$$

This makes G_K into a group as associativity follows from coassociativity as follows: Let $s_1, s_2, s_3 \in G_K$, then

$$\begin{aligned}
(s_1 s_2)s_3 &= [(s_1 \otimes s_2) \circ d \otimes s_3] \circ d \\
&= (s_1 \otimes s_2 \otimes s_3) \circ (d \otimes id) \circ d \\
&= (s_1 \otimes s_2 \otimes s_3) \circ (id \otimes d) \circ d \\
&= (s_1 \otimes ((s_2 \otimes s_3) \circ d)) \circ d \\
&= (s_1 \otimes (s_2 s_3)) \circ d \\
&= s_1(s_2 s_3)
\end{aligned}$$

The properties of ϵ make it a unit, and the map $s \cdot c$ is the inverse to s. To see this we use the properties of the coinverse:

$$\begin{aligned} sc \cdot s &= m(sc \otimes s)d \\ &= m(s \otimes s)(c \otimes id)d \\ &= sm(c \otimes id)d \\ &= si\epsilon \\ &= \epsilon \end{aligned}$$

3.3 Properties of the map i

Proposition. $i : G \to G_K$ is an injective homomorphism

Proof. If $f \in \mathcal{F}(G, K)$ and $d(f) = \sum_j f'_j \otimes f''_j$, then we have

$$f(gh) = \sum_j f'_j(g) f''_j(h)$$

Under the product $s \cdot t$ we get

$$(s \cdot t)(f) = \sum_j s(f'_j) t(f''_j)$$

So to show that i is a homomorphism we need that $i(gh) = i(g)i(h)$:

$$\begin{aligned} (i(g)i(h))(f) &= \sum_j f'_j(g) f''_j(h) \\ &= f(gh) \\ &= i(gh)(f) \end{aligned}$$

Next, let $g \in \ker(i)$, then $f(g) = f(1)$ for any $f \in \mathcal{F}(G, K)$. By a theorem of Peter and Weyl, f separates points, thus $g = 1$. □

Next we endow G_K with a topology. To define the topology we look at the weakest possible topology for which the evaluation maps

$$\lambda_f : G_K \to K \qquad s \mapsto s(f)$$

Are continuous. Here weakest is defined as an alternative name for certain initial topologies:

Definition. Given topological space X, and a family of topological spaces (Y_i) with maps

$$f_i : X \to Y_i$$

Then the initial topology is the coarsest topology $\mathcal{T}$ on X such that

$$f_i : (X, \mathcal{T}) \to Y_i$$

Is continuous

Example. The subspace topology is the initial topology on the subspace with respect to the inclusion map

For the case of $K = \mathbb{R}$ the coarsest such topology is the finite open topology.

The topology has the following important characterization: The maps from an arbitrary topological space X

$$\varphi : X \to G_K$$

Are continuous if and only if the composition $\lambda_f \varphi$ is continuous for all f. Which leads us to the following proposition:

Proposition. For a field K

i) G_K is a topological group

ii) $i : G \to G_K$ is continuous

Proof. ii) This part is straightforward, $i : G \to G_K$ is continuous if and only if $\lambda_f i$ is by the topology characterization, but $\lambda_f i : g \mapsto f(g)$ just an evaluation map, so it's continuous.

i) We need to show that multiplication and inverses composed with λ_f is continuous. Let $\varphi : G_K \times G_K \to G_K$ denote multiplication, then

$$\lambda_f \varphi(s,t) = \lambda_f(s \cdot t) = (s \cdot t)(f) = \sum_j s(f'_j)t(f''_j)$$

So

$$(s,t) \mapsto \sum_j s(f'_j)t(f''_j)$$

And therefore is continuous. This similarly holds for the inverse map $\psi : G_K \to G_K$

$$\lambda_f \psi(s) = \lambda_f(sc) = (sc)(f)$$

So it's also continuous.

□

We can show directly that $G_{\mathbb{R}}$ is a compact Lie group by mapping it into $O(n)$ as a closed subgroup. This is achieved by considering the representation $r : G \to GL_n(\mathbb{K})$ which sends $g \mapsto (r_{ij}(g))$. This induces a continuous homomorphism (we'll show this) between G_K and $GL_n(K)$ which maps $s \mapsto s(r_{ij})$. This map will be important for us moving forward when we talk about the complexification of compact Lie groups.

Proposition. Let $r_K : G_K \to GL_n(K)$ be the map which sends $s \mapsto s(r_{ij})$, then

i) r_K is a continuous homomorphism which makes the following diagram commute

$$\begin{array}{ccc} G & \xrightarrow{\quad i \quad} & G_K \\ & {\scriptstyle r}\searrow \quad \swarrow{\scriptstyle r_K} & \\ & GL_n(K) & \end{array}$$

ii) If the r_{ij} generate $\mathcal{F}(G, K)$ as a K-algebra, then r_K is injective.

iii) If $r : G \hookrightarrow O(n)$, then $r_{\mathbb{R}}(G_{\mathbb{R}}) \subset O(n)$ as a closed subgroup.

Proof. First we prove i. We have that

$$r_K(S \cdot t) = (s \cdot t)(r_{ij}) = \left(\sum_k s(r_{ik} t(r_{kj})) \right) = r_K(s) r_K(t)$$

So it's a homomorphism. By our topology the map r_K is continuous if and only if $\lambda_f r_K$ is continuous, but this is true as this is just the map $s \mapsto s(r_{ij})$. The case of ii follows from the fact that if r_{ij} generate the algebra of representative functions then any element in G_K is completely determined by where it sends r_{ij}.

□

Finally we come to the Tannaka-Krien Duality, or at least half of it. We've shown that G_K is a topological group with the map $i : G \to G_K$ be an injective homomorphism, and in the case of $G_{\mathbb{R}}$ it's a compact Lie group. What remains is to show it's an isomorphism.

Theorem. The map $i : G \to G_{\mathbb{R}}$ is an isomorphism of Lie groups.

Proof. We begin by showing that i induces an isomorphism of the algebra of representative functions. Take $f \in \mathcal{F}(G, \mathbb{R})$, then the evaluation map $\lambda_f : G_{\mathbb{R}} \to \mathbb{R}$ is a representative function. Moreover if f is a matrix coefficient of some representation r, then λ_f is the corresponding coefficient of $r_{\mathbb{R}}$. The map

$$\lambda : \mathcal{F}(G, \mathbb{R}) \to \mathcal{F}(G_{\mathbb{R}}, \mathbb{R})$$

$$f \mapsto \lambda_f$$

is a homomorphism of algebras. If $f \in \mathcal{F}(G, \mathbb{R})$, and $r \in G_{\mathbb{R}}$ then $t \cdot \lambda_f$ maps s to

$$\lambda_f(s \cdot t) = (s \cdot t)(f) = \sum_j s(f'_j) t(f''_j) = \sum_j \lambda_{f'_j}(s) \lambda_{f''_j}(t)$$

Where the last equality is by definition of λ_f. We've demonstrated that the image of λ is $G_{\mathbb{R}}$ invariant. Moreover the λ_f's separates points in $G_{\mathbb{R}}$ so by the Stone-Weierstrauss theorem, $\lambda(\mathcal{F}(G, \mathbb{R}))$ is dense in the sup norm topology. Define the map

$$i^* : \mathcal{F}(G_{\mathbb{R}}, \mathbb{R}) \to \mathcal{F}(G, \mathbb{R})$$

$$s \mapsto si$$

Then the composition of this map with λ is as follows: Let $f \in \mathcal{F}(G, \mathbb{R})$, then

$$i^* \circ \lambda : f \mapsto (\lambda_f \mapsto (\lambda_f i : g \mapsto f(g)))$$

Meaning

$$[(i^* \circ \lambda)(f)](g) = [i^*(\lambda_f)](g) = (\lambda_f i)(g) = f(g)$$

So i^* is a left inverse of λ, similarly one can show that i^* is a right inverse of λ

As $C^0(G_\mathbb{R}, \mathbb{R})$ and $C^0(G, \mathbb{R})$ are the competitions of $\mathcal{F}(G_\mathbb{R}, \mathbb{R}), \mathcal{F}(G, \mathbb{R})$ respectively then

$$i^* : C^0(G_\mathbb{R}, \mathbb{R}) \to C^0(G, \mathbb{R})$$

is an isomorphism as well. Which meas that $i : G \to G_\mathbb{R}$ is a surjective map, therefore together with an earlier result showing this is an injective homomorphism we get the theorem. □

As mentioned this is only half of the theorem. The other half is as follows:

Theorem. Let $(H, m, i, d, \epsilon, c, J)$ be a real commutative skewgroup. Then the map

$$E : H \to \mathcal{F}(H_\mathbb{R}, \mathbb{R})$$

$$E_a\phi = \phi(a)$$

For $a \in H, \phi \in H_\mathbb{R}$, the set of algebra homomorphisms from H to $\mathbb{R}$, is an isomorphism of Hopf Algebras.

For a proof of this see [1]

3.4 The Category Theoretic Perspective

Before moving on we'll very briefly talk about the more much often used Category Theory perspective of duality. The above Hopf algebra approach is one used in [2], [4], and [1], which are the main sources in this project, however those with a background in Category Theory may appreciate this approach more.

Rather than talking about reconstructing G from the algebra of representations, we wish to reconstruct G from its category of representations $\mathbf{Rep}(G)$ over $\mathbb{C}$. To do this we define a forgetful functor as follows:

Definition. A forgetful functor is a functor (morphism between categories) which forgets structure.

The forgetful functor we will use is the functor

$$F : \mathbf{Rep}(G) \to \mathbf{Vect}$$

Which forgets the representation structure, and only remembers the vector space structure of a representation.

Definition. Given categories C and D, and functors $F, G : C \to D$, a natural transformation $\varphi : F \to D$ is a 2-morphism between the functors.

Every element $g \in G$ gives rise to a natural transformation $\varphi(g)$ by defining

$$\varphi(g)_V : F(V) \to F(V)$$

To be multiplication by g whenever V is a representation. Then we can get 3 important properties of $\varphi(g)$:

- It preserves tensor products

$$\varphi(g)_{V \otimes W} = \varphi(g)_V \otimes \varphi(g)_W$$

- It's self conjugate with respect to complex conjugation (working over the ground field $\mathbb{C}$)
- It's the identity map on the trivial representation

Given these we can consider the set of all such natural transformations. $\mathrm{Aut}(F)$, then this object does the job of G_K, and we get

Theorem (Tannaka)**.**

$$G \cong \mathrm{Aut}(F)$$

Krien then expanded on Tannaka's theorem by specifying which categories are of the form $\mathbf{Rep}(G)$ as follows

Theorem (Krein)**.** Let Π be a certain category of finite dimensional linear spaces, equipped with tensor products and involution, which associates with an object in the category the adjoint representation. Then the following conditions are sufficient for there to be a compact group G for which Π is dual.

- There exists an object $I \in \Pi$ such that $I \otimes A \cong A$ for any object $A \in \Pi$
- Every object $A \in Pi$ can be decomposed into a sum of minimal objects
- If A, B are two minimal objects, then $\mathrm{Hom}(A, B)$ is one-dimensional if $A \cong B$, or is 0

If these conditions hold then $\Pi = \mathbf{Rep}(G)$

4 Complexification of Compact Lie Groups

In this section we will work with G_K for $K = \mathbb{C}$. We can show that not only is it a complex Lie group, but we can add some analytic structure to make it a compact analytic group. Even more things we can say about $G_{\mathbb{C}}$ is that it's an algebraic group, which means it's an affine variety with a group structure.

We begin with the definition of Complexification:

Definition. The complexification of a compact Lie group G is the pair $(G_{\mathbb{C}}, i)$ where i is the homomorphism defined above:

$$i : G \to G_{\mathbb{C}}$$

We've already seen in class some examples of various complexifications, even if we didn't know it.

Example. $SL_n(\mathbb{C})$ is the complexification of $SL_n(\mathbb{R})$

Example. $U(n) \subset GL_n(\mathbb{C})$ is a complexification.

Example. $SU(n) \subset SL_n(\mathbb{C})$ is a complexification

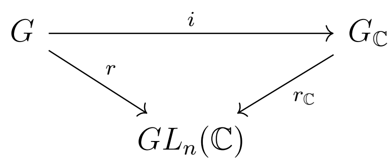
Other examples of complexification include the following

Example. $O(n) \subset O(n, \mathbb{C})$ and $SO(n) \subset SO(n, \mathbb{C})$

Proposition.

$$L(G) \otimes_{\mathbb{R}} \mathbb{C} \cong L(G_{\mathbb{C}})$$

To show these are indeed complexifications we need to use the following commutative diagram that we've seen before



We can demonstrate that this $r_{\mathbb{C}}$ is a holomorphic representation. First we can show a surprising proposition:

Proposition. If $r \to GL_n\mathbb{C}$ is a representation such that r_{kj} generate $\mathcal{F}(G, \mathbb{C})$, then $r_{\mathbb{C}} : G_{\mathbb{C}} \to V(I) \subset GL_n(\mathbb{C})$ is a bijection. Here $V(I)$ is the affine variety of the ideal I which is the kernel of the map

$$\mathbb{C}[X_{kj}] \to \mathcal{F}(G, \mathbb{C})$$

$$X_{kj} \mapsto r_{kj}$$

Proof. Let r be a faithful representation such that the entries r_{kj} of r generate $\mathcal{F}(G, \mathbb{C})$ as a $\mathbb{C}$-algebra. Then

$$\mathcal{F}(G, \mathbb{C}) \cong CX_{kj}/I$$

We can get a bijection

$$\sigma : V(I) \to G_{\mathbb{C}}$$

given by

$$z \mapsto \sigma_z : (p + I \to p(z))$$

When we compose this map with $r_{\mathbb{C}}$ we see

$$z \mapsto (\sigma_z \mapsto (\sigma_z(r_{kj})) = X_{kj}(z) = z)$$

So it's the identity map, and the following diagram commutes

$$\begin{array}{ccc} V(I) & \xrightarrow{\sigma} & G_{\mathbb{C}} \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle r_{\mathbb{C}}} \\ GL_n\mathbb{C} & \xrightarrow{id} & GL_n(\mathbb{C}) \end{array}$$

□

As $V(I)$ is a closed subgroup within $GL_n(\mathbb{C})$ it's a Lie subgroup.

Let $P(n)$ be the set of positive definite Hermitian $n \times n$ matrices then multiplication of an element in $P(n)$ with an element in $U(n)$ is a homeomorphism:

$$U(n) \times P(n) \to GL_n(\mathbb{C})$$

$$(H, P) \mapsto HP$$

Proposition. Let $\tilde{G} = r_{\mathbb{C}}(G_{\mathbb{C}})$

i) If we express $A \in \tilde{G}$ as the product $A = HP$ as above, then $H, P \in \tilde{G}$ and we get a multiplication map

$$(\tilde{G} \cap U(n)) \times (\tilde{G} \cap P(n)) \to \tilde{G}$$

$$(H, P) \mapsto HP$$

which is a homeomorphism

ii) $\tilde{G} \cap P(n)$ is homeomorphic to a Euclidean space of dimension $\dim \tilde{G} \cap U(n)$

iii) $\tilde{G} \cap U(n)$ is a maximal compact subgroup of $\tilde{G}$

Proof. A full proof is found in [2]. We prove parts *ii* and *iii*
For *ii* we show that the map $X \mapsto \exp(iX)$ is a homeomorphism between the Lie algebra of $\tilde{G} \cap U(n)$ and $\tilde{G} \cap P(n)$. Let $X \in L(\tilde{G} \cap U(n))$, then $\exp(tX) \in \tilde{G} \cap U(n)$ for all real t, then writing $\tilde{G}$ as the variety $V(I)$ we know that any polynomial in I has $\exp(tX)$ as a zero for $t \in \mathbb{R}$. As such the analytic function $t \mapsto p(\exp(itX))$ vanishes, and so $\exp(itX) \in \tilde{G}$. As the Lie algebra of $\tilde{G} \cap U(n)$ lies in the Lie algebra of $U(n)$ the matrix itX is Hermitian, and therefore $\exp(itX) \in P(n)$. Hence the Lie algebra of $\tilde{G}$ is the direct sum of the subspaces $L = L(\tilde{G} \cap U(n))$ and iL. As the exp map above is the restriction of the homeomorphism $X \mapsto \exp(iX)$ from the set of skew Hermitian $n \times n$ matrices to $P(n)$ we get the desired result.

For *iii* we see that $\tilde{G} \cap U(n)$ is a closed subgroup of the compact group $U(n)$, thus it's compact. Assume for contradiction that there is another compact subgroup K which is maximal. In this case it would contain an element of $\tilde{G} \cap P(n)$ other than the identity matrix, this can be see by recalling that we have a homeomorphism from i to $\tilde{G}$. But by *ii* by dimension considerations this cannot happen.

□

We return to the following proposition

Proposition.

$$L(G) \otimes_{\mathbb{R}} \mathbb{C} \cong L(G_{\mathbb{C}})$$

Proof. We have that $L(\tilde{G}) = L(\tilde{G} \cap U(n)) \oplus iL(\tilde{G} \cap U(n))$. As $r_{\mathbb{R}}(G_{\mathbb{R}}) = \tilde{G} \cap U(n)$ then it's isomorphic to G via the discussion in the Tannaka-Krien section. As such $L(\tilde{G} \cap U(n)) \cong L(G)$ as Lie algebras, and we get

$$L(G) \otimes_{\mathbb{R}} \mathbb{C} \cong L \otimes_{\mathbb{R}} \mathbb{C} \cong L \oplus iL \cong L(G_{\mathbb{C}})$$

Which preserves Lie brackets. □

Proposition. Given a representation $r \to GL_n(\mathbb{C})$ there is a unique holomorphic representation $r_{\mathbb{C}} : G_{\mathbb{C}} \to GL_n(\mathbb{C})$ such that $r_{\mathbb{C}} \circ i = r$

Proof. Recall that we have $r_{\mathbb{C}}(s) = (s(r_{kj}))$, then if we have

$$\mathcal{F}(G, \mathbb{C}) = \mathbb{C}[a_1, ..., a_d] \cong \mathbb{C}[X_1, ..., X_D]/I$$

We know that $G_{\mathbb{C}}$ maps bijectively onto $V(I)$ via $r_{\mathbb{C}}$, and as such each entry r_{kj} is an algebraic function, and so

$$r_{\mathbb{C}} : V(I) \subset \mathbb{C}^d \to GL_n(\mathbb{C}) \subset \mathbb{C}^{m^2}$$

is a holomorphic map, because it's algebraic.

Recall that $\tilde{G} \cong \tilde{G} \cap U(n) \times \tilde{G} \cap P(n)$. If we're given the values of a holomorphic representation on $\tilde{G} \cap U(n)$ then for any element in $\tilde{G} \cap P(n)$ of the form $\exp(iX)$ we can determine the values of that representation on $\exp(tX)$ for all $t \in \mathbb{R}$. □

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